

AD 663877

THE
TECHNOLOGICAL
INSTITUTE



DDC
RECEIVED
JAN 19 1968
RECEIVED
B

VB

This document has been approved
for public release and sale; its
distribution is unlimited.

17



NORTHWESTERN UNIVERSITY

EVANSTON, ILLINOIS

Reproduced by the
CLEARINGHOUSE
for Federal Scientific & Technical
Information Springfield Va. 22151

The Technological Institute The College of Arts and Sciences
Northwestern University

AN EXPLICIT SOLUTION OF A SPECIAL
CLASS OF LINEAR PROGRAMMING
PROBLEMS

by

A. Ben-Israel
A. Charnes

Northwestern University
Evanston, Illinois

December 1967

Part of the research underlying this report was undertaken for the Office of Naval Research, Contract NONR-1228(10), Project NR 047-021, for the U. S. Army Research Office - Durham, Contract No. DA-31-124-ARO-D-322, and for the National Science Foundation, Project GP 7550 at Northwestern University. Reproduction of this paper in whole or in part is permitted for any purpose of the United States Government.

SYSTEMS RESEARCH GROUP

A. Charnes, Director

Introduction

The linear programs considered here are of the form:

$$\begin{aligned} & \text{Maximize } (c, x) \\ \text{(LP)} \quad & \text{subject to} \\ & a \leq Ax \leq b \end{aligned}$$

where A is of full row rank, and (LP) is feasible with bounded optimal solutions.

The main result, equation (24), is an explicit representation of the general optimal solution of (LP), in terms of a generalized inverse of $A^{1/}$. This explicit solution of (LP) -explicit in the sense that $A^{-1}b$ is an explicit solution of $Ax = b$ - has obvious theoretical (and possibly computational) advantages over the well known iterative methods of linear programming, e. g. [5], [8].

The results are illustrated by a simple example, and extensions to general linear programs are discussed.

Preliminaries and notations

We denote by:

R^n the n -dimensional real vector space

For any two vectors x, y in R^n :

$x \geq y$ denotes $x_i \geq y_i$ ($i = 1, \dots, n$)

^{1/} For other applications of generalized inverses in linear programming see [14], [7] and [6].

For any subspace L of R^n :

L^\perp denotes the orthogonal complement of L

P_L denotes the perpendicular projection on L

For any $m \times n$ real matrix A :

A^T -- the transpose of A

$R(A)$ -- the range space of A

$N(A)$ -- the null space of A .

For the fixed $m \times n$ real matrix A consider the 4 matrix equations:

- (1) $AXA = A$
- (2) $XAX = X$
- (3) $(AX)^T = AX$
- (4) $(XA)^T = XA$

We denote by $A\{i, j, \dots, k\}$ the set of $n \times m$ real matrices X satisfying equations (i), (j), ..., (k), ($1 \leq i, j, \dots, k \leq 4$). These sets $A\{i, j, \dots, k\}$, ($1 \leq i, j, \dots, k \leq 4$), are nonempty because $A\{1, 2, 3, 4\}$ is nonempty, e.g. [13].

A matrix $X \in A\{i, j, \dots, k\}$ is called an $\{i, j, \dots, k\}$ - g.i. (generalized inverse) of A . The $\{1, 2, 3, 4\}$ - g.i. of A is unique, and is the well known Moore-Penrose generalized inverse, e.g. [13], [12], denoted by A^+ . For some applications a weaker g.i. will do, e.g. [3], [4], [11] and [10]. Thus for solving linear equations (and for the purpose of this paper) $\{1\}$ - g.i.'s are sufficient, as shown by the following:

Lemma 1 ([3], [13]): The linear equations

$$(5) \quad Ax = b$$

are solvable iff for any $T \in A\{1\}$

(6) $ATb = b,$

in which case the general solution of (5) is:

(7) $x = Tb + (I - TA)y, y \text{ arbitrary}$ □

The set $A\{1\}$ is represented in terms of one of its elements as follows:

Lemma 2 ([3]): Let R be any $\{1\}$ - g. i. of A . Then

(8) $A\{1\} = \{RAR + Y - RAYAR : Y \text{ arbitrary}$

$\text{nxm real matrix} \}$

Projections associated with g. i.'s are given in: □

Lemma 3 ([1]):

(a) If $S \in A\{1,3\}$ then

(9) $AS = P_{R(A)}$

(b) If $T \in A\{1,2,4\}$ then

(10) $TA = P_{R(A^T)}$

A recipe for computing a $\{1\}$ - g. i. and for constructing a basis of $N(A)$ is given in:

Lemma 4: Let A be an $m \times n$ real matrix of rank r , and let E be a nonsingular $m \times m$ real matrix such that:

$$(11) \quad EA = \left(\begin{array}{c|c} I_r & \Delta \\ \hline O_{(m-r) \times n} & \end{array} \right) P$$

where P is a permutation matrix.

Conclusions:

(a) Let E be partitioned

$$(12) \quad E = \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array} \right) \quad \text{where } E_{11} \text{ is } r \times r$$

then the $n \times m$ matrix

$$(13) \quad T = P^T \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline O_{(n-r) \times m} & \end{array} \right)$$

is a $\{1,2\}$ - inverse of A .

(b) The columns of the $n \times (n-r)$ matrix

$$(14) \quad N = P^T \left(\begin{array}{c} -\Delta \\ \hline I_{n-r} \end{array} \right)$$

form a basis of $N(A)$.

Proof:

(a) Consider the $n \times n$ nonsingular matrix

$$(15) \quad F = P^T \left(\begin{array}{c|c} I_r & -\Delta \\ \hline O & I_{n-r} \end{array} \right)$$

From (11), (15) and $PP^T = I$ we get

$$EAF = \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right)$$

or

$$(16) \quad A = E^{-1} \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right) F^{-1}$$

Now the matrix

$$(17) \quad T = F \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right) E$$

is a $\{1, 2\}$ - g. i. of A , as shown by substituting (16) and (17) in (1) and (2).

Substituting (12) and (15) in (17) we get

$$\begin{aligned} T &= P^T \left(\begin{array}{c|c} I_r & -A \\ \hline O & I_{n-r} \end{array} \right) \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right) \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array} \right) \\ &= P^T \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline O & O \end{array} \right) \end{aligned}$$

which proves (13).

(b) Obvious from (11) and the facts : $PP^T = I$ and E nonsingular. \square

For other results and references on $\{1\}$ - g. i.'s see [4], [10] and [1].

Results:

Consider the linear programming problem:

(LP) Maximize (c, x)
subject to

$$(18) \quad a \leq Ax \leq b,$$

for given

$$A = (a_{ij}), \quad a = (a_i), \quad b = (b_i), \quad c = (c_j)$$

$$(i = 1, \dots, m; \quad j = 1, \dots, n),$$

and assume:

Assumption 1: (LP) is feasible, i. e.

$$(19) \quad S = \{x \in R^n : a \leq Ax \leq b\} \neq \emptyset$$

The following properties of S are obvious:

Lemma 5: If (19) then

$$(20) \quad S = S + N(A) = \{x + y : x \in S, y \in N(A)\}$$

and the set

$$(21) \quad P_{R(A^T)} S = \{P_{R(A^T)} x : x \in S\} \text{ is bounded.}$$

□

The case when (LP) has a finite maximum is characterized in:

Lemma 6: (LP) has a bounded optimal solution iff

$$(22) \quad c \perp N(A)$$

Proof: If: From (22) and the fact

$$N(A) = R(A^T)^\perp$$

it follows that

$$\{(c, x) : x \in S\} = \{(c, x) : x \in P_{R(A^T)} S\}$$

and is a bounded interval, since (21) is a bounded set.

Only if: Suppose $P_{N(A)} c \neq 0$. Then the interval

$$\{(c, x) : x \in S\}$$

is the entire real line, by (20)

□

We make now 2 additional assumptions:

Assumption 2: (LP) has a bounded optimal solution, i. e. (22) holds.

Assumption 3: The matrix A is of full row rank:

$$(23) \quad \text{rank } A = m$$

An explicit representation of the general optimal solution of (LP) is now given:

Theorem: Let (LP) satisfy assumptions 1, 2 and 3, and let the $n \times m$ matrix

T with columns (t_1, t_2, \dots, t_m) be a $\{1\}$ -g.i. of A. Then the optimal solutions of (LP) form the manifold:

$$(24) \quad x = \sum_{i \in I_-} t_i a_i + \sum_{i \in I_+} t_i b_i + \sum_{i \in I_0} t_i (\theta_i b_i + (1 - \theta_i) a_i) + N(A)$$

where

$$(25) \quad \begin{aligned} I_- &= \{i : (c, t_i) < 0\} \\ I_+ &= \{i : (c, t_i) > 0\} \\ I_0 &= \{i : (c, t_i) = 0\} \end{aligned}$$

and $0 \leq \theta_i \leq 1, \quad i \in I_0.$

Proof: From (23) it follows that

$$R(A) = R^m$$

so that for any z in R^m we have

$$(26) \quad z = A x$$

where

$$(27) \quad x = T z + N(A), \text{ by lemma 1.}$$

Substituting (26), (27) in (LP) we conclude from (22) that (LP) is

equivalent to the following linear program over a parallelopiped:

(LPP) Maximize (c, Tz)
 subject to

$$(28) \quad a \leq z \leq b$$

whose optimal solution is obviously:

$$z_i = \begin{cases} a_i & i \in I_- \\ b_i & i \in I_+ \\ \theta_i b_i + (1 - \theta_i) a_i & i \in I_0 \end{cases}$$

for any $0 \leq \theta_i \leq 1$, $i \in I_0$.

From (27) and (29) it follows that the manifold (24) is the set of optimal solutions of (LP). . □

Remark: It can be shown directly that the set (24) is independent of the particular $\{1\}$ - g. i. used in its definition. We need:

Lemma 7: Let A be an $m \times n$ matrix of rank m , and let T be any $\{1\}$ - g. i. of A. Then

$$(30) \quad T = A^+ + W$$

where W is a matrix whose columns lie in $N(A)$.

Proof: Lemma 2, with $R = A^+$, the unique $\{1, 2, 3, 4\}$ - g. i. of A, gives:

$$(31) \quad A\{1\} = A^+ A A^+ + Y - A^+ A Y A A^+ , \quad Y \text{ arbitrary}$$

$$= A^+ + P_{N(A)} Y , \quad Y \text{ arbitrary}$$

This follows from $A^+AA^+ = A^+$ by (2),

$$\begin{aligned} AA^+ &= P_{R(A)} \quad , \text{ by (9) ,} \\ &= I \quad , \text{ since rank } A = m \end{aligned}$$

and

$$\begin{aligned} I - A^+A &= I - P_{R(A^T)} \quad , \text{ by (10) ,} \\ &= P_{N(A)} . \end{aligned}$$

Now (30) follows from (31) with $W = P_{N(A)} Y$. □

From lemma 7 and (22) it follows that the sets I_- , I_+ , I_0 defined by (25) and the general solution (24) are independent of the $\{1\}$ - g. i. used in the theorem.

Example:

The problem, of class (LP) , is:

$$(32) \quad \text{Maximize } 2x_1 - x_2 - x_3 + 3x_4$$

subject to:

$$\begin{aligned} 0 &\leq x_1 + 2x_2 - x_3 && \leq 1 \\ -3 &\leq -x_1 + x_3 - x_4 && \leq 0 \\ 1 &\leq 2x_1 + x_2 - 3x_3 + x_4 && \leq 3 \end{aligned}$$

We use lemma 4 to compute a $\{1\}$ - g. i. of

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -3 & 1 \end{pmatrix}$$

by diagonalizing the $m \times (n + m)$ matrix

$$A^{(0)} = (A, I_m) :$$

$A^{(0)} =$	$\begin{array}{cccc ccc} \textcircled{1} & 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 1 & 0 & 0 & 1 \end{array}$	pivots are circled
$A^{(1)} =$	$\begin{array}{cccc ccc} 1 & 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 0 & -1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 & 1 \end{array}$	
$A^{(2)} =$	$\begin{array}{cccc ccc} 1 & 0 & -1 & 1 & 0 & -5/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & \textcircled{-1} & -1/2 & -1/2 & -3/2 & -1 \end{array}$	
$A^{(3)} =$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 3/2 & 1/2 & -5/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -3/2 & -1 \end{array}$	

Assumption 3 is satisfied if the last matrix $A^{(m)}$ is of the form

$$A^{(m)} = ((I_m, \Delta) P \mid E)$$

Indeed from $A^{(3)}$ we read

$$(33) \quad \Delta = \begin{pmatrix} 3/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad P = I, \quad E = \begin{pmatrix} 1/2 & -5/2 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -3/2 & -1 \end{pmatrix}$$

and by (14) it follows that $N(A)$ is spanned by the vector

$$(34) \quad N = \begin{pmatrix} -3/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

and that assumption 2 is satisfied for $c = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 3 \end{pmatrix}$.

From (13) and (33) we get a $\{1\}$ - g. i. of A :

$$(35) \quad T = \begin{pmatrix} 1/2 & -5/2 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -3/2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

for which

$$c^T T = (2, -1, -1, 3) \begin{pmatrix} 1/2 & -5/2 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -3/2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = (0, -4, -1)$$

so that

$$(36) \quad I_- = \{2, 3\}, \quad I_+ = \emptyset, \quad I_0 = \{1\}.$$

The general optimal solution of (32) is from (24), (34), (35) and

(36):

$$(37) \quad x = \theta_1 \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} -5/2 \\ 1/2 \\ -3/2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

with $0 \leq \theta_1 \leq 1$

and λ arbitrary.

The optimal value of $(c, x) = 11$.

A different choice of pivots in $A^{(i)}$ ($i = 0, 1, 2$) could result in different matrices in (33) and (35), but the sets (36) and the manifold (37) are unchanged.

Discussion

Linear programs arising in concrete applications are usually feasible and possess bounded optimal solutions. Therefore assumptions 1 and 2 are not too restrictive.

Also any linear program with inequality constraints can be rewritten as our problem (LP), by setting the missing a_i and b_i as: $-M$ and $+M$ respectively, where $M > 0$ is a sufficiently large number. (If M appears in an optimal solution, then the problem has unbounded optimal solutions).

The remaining assumption 3 is a true restriction on the scope of our method. It is typically violated by linear programs of the form:

Maximize (c, x)
subject to

$$(38) \quad \begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

which are rewritten as our (LP) :

Maximize (c, x)

$$(39) \quad \begin{pmatrix} -Me \\ 0 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} b \\ Me \end{pmatrix}$$

where $M > 0$ is sufficiently large, and $e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

There are several ways of applying our method to problems (LP) without assuming A to have full row rank. One possibility is to partition (LP) as follows:

Max (c, x)

$$(40) \quad \begin{aligned} a^1 &\leq A^1 x \leq b^1 \\ a^2 &\leq A^2 x \leq b^2 \end{aligned}$$

where A^1 is an rxn submatrix of the mxn matrix A and

$$(41) \quad \text{rank } A^1 = \text{rank } A = r.$$

From (41) it follows that

$$(42) \quad N(A) = N(A^1)$$

and by lemma 6, problem (40) has a bounded optimal solution iff

$$(43) \quad c \perp N(A^1) .$$

The subproblem

$$(44) \quad \begin{aligned} &\text{Max } (c, x) \\ &a^1 \leq A^1 x \leq b^1 \end{aligned}$$

thus satisfies our assumptions, and (24) can be used to obtain the manifold

$$(45) \quad x^1 + N(A)$$

of optimal solutions of (44).

Any vector in (45) which satisfies the remaining constraints

$$a^2 \leq A^2 x \leq b^2$$

of (40) is clearly an optimal solution of (40). In the absence of such a vector, an optimal solution of (40) can be found in a finite number of iterations, where at each iteration A^1 is changed by one row in an obvious manner. Our method may thus serve as a start for a dual simplex method.

Another possibility is to partition (LP) into k subproblems

$$\text{Max } (c, x^i)$$

$$(46. i) \quad a^i \leq A^i x^i \leq b^i \quad (i = 1, \dots, k)$$

where each subproblem satisfies our assumptions 1 and 3. Since assumption 2 is assumed for (LP), we solve (46. i) by (24) as if $c \perp N(A^i)$, $i = 1, \dots, k$. The resulting optimal solutions are generally different, but may be forced to coincide in a finite number of iterations of the decomposition method of Dantzig-Wolfe [9]. These results are contained in [15].

REFERENCES

- [1] A. Ben-Israel, "Lectures on generalized inverses," Northwestern University, Spring 1967
- [2] A. Ben-Israel and A. Charnes, "Contributions to the theory of generalized inverses," J. Soc. Indust. Appl. Math. 11(1963), 667-699
- [3] A. Bjerhammar, "Applications of calculus of matrices to method of least squares; with special reference to geodetic calculations," Kungl. Tekn. Högsk. Handl. No. 49, (1951), 86 pp. (Reviewed in Math. Rev. 14-1127).
- [4] A. Bjerhammar, "A generalized matrix algebra," Kungl. Tekn. Högsk. Handl. No. 124 (1958), 32 pp. (Reviewed in Math. Rev. 20 #7038)
- [5] A. Charnes and W.W. Cooper, Management models and industrial applications of linear programming, Vols. I, II, J. Wiley 1961
- [6] A. Charnes and M. Kirby, "Modular designs, generalized inverses, and convex programming," Operations Research 13(1965), 836-847
- [7] R.E. Cline, "Representations for the generalized inverse of matrices with applications in linear programming," Doctoral thesis, Purdue University, 1963.
- [8] G.B. Dantzig, Linear programming and extensions, Princeton University Press, 1963
- [9] G.B. Dantzig and P. Wolfe, "Decomposition principle for linear programs," Operations Research 8 (1960), 101-111
- [10] A.G. Fisher, "On construction and properties of the generalized inverse," SIAM J. Appl. Math. 15 (1967), 269-272
- [11] A.J. Goldman and M. Zelen, "Weak generalized inverses and minimum variance linear unbiased estimation," J. Res. Nat. Bur. Standards Sect. B 68(1964), 151-172
- [12] T.N.E. Greville, "Some applications of the pseudoinverse of a matrix," SIAM Rev. 2(1960), 15-22
- [13] R. Penrose, "A generalized inverse for matrices," Proc. Cambridge Philos. Soc. 51(1955), 406-413
- [14] L.D. Pyle, "The generalized inverse in linear programming," Doctoral thesis, Purdue University, 1963
- [15] P.D. Roberts and A. Ben-Israel, "On explicit solutions of linear programming," (forthcoming)

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Northwestern University		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP Not Applicable	
3. REPORT TITLE AN EXPLICIT SOLUTION OF A SPECIAL CLASS OF LINEAR PROGRAMMING PROBLEMS			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report			
5. AUTHOR(S) (Last name, first name, initial) Adi Ben-Israel and Abraham Charnes			
6. REPORT DATE December 1967		7a. TOTAL NO. OF PAGES 14	7b. NO. OF PAGES 15
8a. CONTRACT OR GRANT NO. NONR-1228(10) - Project NR 047-021		9a. ORIGINATOR'S REPORT NUMBER(S) Systems Research Memorandum No. 191	
b. XXXXXXXX Contract No. DA-31-124-ARO-D-322			
c. NSF Contract Project GP 7550		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.			
10. AVAILABILITY/LIMITATION NOTICES Distribution of this document is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch, Office of Naval Research	
13. ABSTRACT The linear programs considered here are of the form: $\begin{array}{ll} \text{Maximize (c, x)} \\ \text{(LP)} & \text{subject to} \\ & a \leq Ax \leq b \end{array}$ where A is of full row rank, and (LP) is feasible with bounded optimal solutions. The main result, equation (24), is an explicit representation of the general optimal solution of (LP), in terms of a generalized inverse of A . This explicit solution of (LP) - explicit in the sense that $A^{-1}b$ is an explicit solution of $Ax = b$ - has obvious theoretical (and possibly computational) advantages over the well known iterative methods of linear programming. The results are illustrated by a simple example, and extensions to general linear programs are discussed.			

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Linear Programming						

INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those

imposed by security classification, using standard statements such as:

- (1) "Qualified requesters may obtain copies of this report from DDC."
- (2) "Foreign announcement and dissemination of this report by DDC is not authorized."
- (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."
- (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."
- (5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical content. The assignment of links, roles, and weights is optional.